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Bounds for the weighted Lebesgue functions for Freud weights on a larger interval

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Abstract

The bounds of the weighted Lebesgue functions for Freud weights between the largest and the smallest zeros of orthonormal polynomials are established by means of the extension of the Erdős–Turán inequality for the sum of successive fundamental polynomials of Lagrange interpolation. An extension to a larger interval is also done.

Keywords: Lebesgue function; Freud weight; Orthogonal polynomials; Erdős–Turán inequality; Lagrange interpolation

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1. Introduction and results

We consider $W := \mathrm{e}^{-Q}$, where $Q:\mathbb{R} \to \mathbb{R}$ is even and continuous in $\mathbb{R}$, $Q' > 0$ in $(0, \infty)$, Q'' is continuous in $(0, \infty)$, while for some $A, B > 1$,

$$A \leqslant \left[\frac{\mathrm{d}}{\mathrm{d}x}(xQ'(x))\right] \Big/ Q'(x) \leqslant B, \quad x \in (0, \infty). \tag{1.1}$$

We call such W a *Freud weight*. For example

$$W_\beta := \exp(-|x|^\beta), \quad \beta > 1,$$

is one such a weight.

Corresponding to W^2 is a sequence of orthonormal polynomials $\{p_n(x)\}$, where

$$p_n(x) := p_n(W^2, x) = \gamma_n x^n + \cdots,$$

is the nth orthonormal polynomial of W^2 and $\gamma_n > 0$ is its leading coefficient. The zeros of $p_n(x)$ will be denoted by

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < +\infty \tag{1.2}$$

arranged in increasing order.

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We define for $\alpha \geqslant 0$, the nth *weighted Lebesgue function* for the Freud weight $W = e^{-Q}$ by

$$\Lambda_n(x) := W(x) \sum_{k=1}^{n} |l_{kn}(x)| W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha}, \tag{1.3}$$

where

$$l_{kn}(x) := \frac{p_n(x)}{p_n'(x_{kn})(x - x_{kn})} \tag{1.4}$$

are the fundamental polynomials associated with W^2.

The significance of the bounds of $\Lambda_n(x)$ lies in the study of convergence of Lagrange interpolation at the zeros of orthogonal polynomials to continuous functions $f: \mathbb{R} \to \mathbb{R}$ satisfying the decay condition

$$\lim_{|x| \to \infty} |f(x)| W(x)(1 + |x|)^{\alpha} = 0, \quad \text{for } \alpha \geqslant 0. \tag{1.5}$$

The bounds of $\Lambda_n(x)$ were established in [6, 7] with a full discussion of applications to uniform and pointwise convergence of Lagrange interpolation at the zeros of orthogonal polynomials to functions characterized by (1.5) in the former. Another work on the bounds of Lebesgue functions and their applications to Lagrange interpolation can be obtained in [1, 2, 8]. In [6] we determined the bounds of $\Lambda_n(x)$ on a small interval. In particular we proved:

Theorem 1.1. *Let $\sigma \in (0, 1)$ be fixed and $\alpha \geqslant 0$. Define*

$$\log^* n := \begin{cases} \log n, & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha \neq 1. \end{cases}$$

Then uniformly for $|x| \leqslant \sigma a_n$,

$$\Lambda_n(x) \sim (1 + |x|)^{-\alpha} + \sqrt{a_n} |p_n(W^2, x)| W(x) \{(1 + |x|)^{-\alpha} \log n + (1 + |x|)^{-\hat{\alpha}} \log^* n\}. \tag{1.6}$$

However we could not extend (1.6) to a larger interval due to our inability to determine suitable lower bounds on larger set. Our failure to obtain a suitable lower bound in this respect can be attributed to lack of a more general form of the Erdős–Turán inequality (cf. [4]) for the sum of consecutive fundamental polynomials.

The primary objective of this paper is to extend (1.6) to the interval $[x_{nn}, x_{1n}]$ (recall (1.2)) and even a larger interval.

Notation. To state out result we need some notation:

(1) Throughout, $L, C, C_1, C_2, \ldots$ are positive constants independent of n and $x \in \mathbb{R}$. The same symbol does not necessarily denote the same constant in different occurrences.

(2) We use $\sim$ in the following sense: If A and B are two expressions depending on some variables and indices then

$$A \sim B \quad \Leftrightarrow \quad |AB^{-1}| \leqslant C_1 \text{ and } |A^{-1}B| \leqslant C_2.$$

(3) For $u > 0$, the uth *Mhaskar–Rahmanov–Saff* a_u is the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)/\sqrt{1-t^2}\,\mathrm{d}t. \tag{1.7}$$

(4) For $x \in \mathbb{R}$, define

$$\psi_n(x) := \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}, \quad n \geqslant 1. \tag{1.8}$$

(5) For $x \in \mathbb{R}$, let $x_{k(x),n}$ denote a zero of $p_n(W^2, \xi)$ closest to x. Let

$\mathscr{S} := \{x_{k(x),n}\colon x \in \mathbb{R}\}$

be the set containing all the zeros closest to x, and define

$$\tilde{\Lambda}_n(x) := W(x) \sum_{x_{kn} \in \mathscr{S}} |l_{kn}(x)| W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha}. \tag{1.9}$$

(6) Define $x_{0n} := x_{1n}(1 + n^{-1/2})$ and $x_{n+1,n} := x_{nn}(1 + n^{-1/2})$.

Our result is:

Theorem 1.2. *For* $\alpha \geqslant 0$. *Define*

$$\log^* n := \begin{cases} \log n, & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha \neq 1. \end{cases}$$

(a) *Then uniformly for* $x \in [x_{nn}, x_{1n}]$,

$$\begin{aligned} \Lambda_n(x) &:= W(x) \sum_{j=1}^{n} |l_{kn}(x)| W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha} \\ &\sim (1 + |x|)^{-\alpha} + \sqrt{a_n}|p_n(W^2, x)| W(x)\{(1 + |x|)^{-\alpha}\log n + (1 + |x|)^{-\hat{\alpha}}\log^* n\}. \end{aligned} \tag{1.10}$$

(b) (1.10) *holds uniformly for* $|x| \leqslant a_n(1 + Ln^{-2/3})$, *for any fixed* $L > 0$.

2. Preliminary results

The proof of the main result is a consequence of a number of lemmas.

Lemma 2.1. (a) *For* $n \geqslant 1$,

$$\left|\frac{x_{1n}}{a_n} - 1\right| \leqslant Cn^{-2/3} \tag{2.1}$$

and uniformly for $n \geqslant 3$ *and* $1 \leqslant k \leqslant n$,

$$x_{k-1,n} - x_{k+1,n} \sim \frac{a_n}{n}\psi_n(x_{kn})^{-1/2}. \tag{2.2}$$

(b) *Uniformly for* $n \geqslant 1$, $1 \leqslant k \leqslant n$, *and* $x \in \mathbb{R}$,

$$|l_{kn}(x)|\, W^{-1}(x_{kn}) W(x) \leqslant C. \tag{2.3}$$

(c) *If* A, B *are as in* (1.1), *then*

$$u^{1/B} \leqslant a_u/a_1 \leqslant u^{1/A}. \tag{2.4}$$

(d) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and* $x \in \mathbb{R}$,

$$|l_{jn}(x)| \sim (a_n^{3/2}/n) W(x_{jn}) \psi_n(x_{jn})^{-1/4} \frac{|p_n(x)|}{|x - x_{jn}|}. \tag{2.5}$$

(e) *There exists* $C_1 > 0$, *such that uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and*

$$|x - x_{jn}| \leqslant C_1 \frac{a_n}{n} \psi_n(x_{jn})^{-1/2},$$

we have

$$|p_n(x)|\, W(x) \sim (n/a_n^{3/2}) \psi_n(x_{jn})^{1/4} |x - x_{jn}|. \tag{2.6}$$

Proof. (a) This is Corollary 1.2(a) in [3]. (b) This is Lemma 2.6(b) in [5]. (c) This is Lemma 5.2(b) in [3]. (d) This is Lemma 2.6(a) in [5]. (e) This is Lemma 2.6(c) in [5]. □

Lemma 2.2. *Let* $\hat{\Lambda}_n(x) := \Lambda_n(x) - \tilde{\Lambda}_n(x)$. *Then*

$$\hat{\Lambda}_n(x) \sim \sqrt{a_n}\, |p_n(W^2, x)|\, W(x) \{(1 + |x|)^{-\alpha} \log n + (1 + |x|)^{-\hat{\alpha}} \log^* n\}.$$

Proof. This is Lemma 2.10 in [7]. □

The following results in the extension of the Erdős–Turán inequality.

Lemma 2.3. *Let* $W = e^{-Q}$ *be a Freud weight. Then for* $\beta > 0$, *and* $1 \leqslant k \leqslant n - 1$, *and* $x \in [x_{k+1,n}, x_{kn}]$,

$$W^{\beta}(x) l_{kn}(x) W^{-\beta}(x_{kn}) + W^{\beta}(x) l_{k+1,n}(x) W^{-\beta}(x_{k+1,n}) \geqslant 1. \tag{2.7}$$

Proof. This is Theorem 1 in [4]. □

3. Proof of Theorem 1.1

Now we estimate $\tilde{\Lambda}_n(x)$. Let $x_{k(x),n}$ denote the closest abscissa to $x \in \mathbb{R}$. We can assume that $x \geqslant 0$. Observe that because of the spacing (2.2), $\tilde{\Lambda}_n(x)$ has a finite number of terms.

Using (2.3) we obtain

$$\begin{aligned} \tilde{\Lambda}_n(x) &\leqslant C \sum_{x_{kn} \in \mathscr{S}} (1 + |x_{kn}|)^{-\alpha} \\ &\leqslant C_1 (1 + |x|)^{-\alpha}, \end{aligned} \tag{2.8}$$

which is an easy consequence of (2.2): From (2.2) for $2 \leqslant k \leqslant n-1$,

$$1+|t| \sim 1+|x_{kn}|, \quad t \in [x_{k+1,n}, x_{k-1,n}]. \tag{2.9}$$

Assume for simplicity that $k = k(x)$ (if not, $x \in [x_{kn}, x_{k-1,n}]$ and the argument is similar). Now it is known that if $x \in [x_{k+1,n}, x_{kn}]$, for some $1 \leqslant k \leqslant n-1$, then by the generalized Erdős–Turán inequality (2.7) with $\beta = 1$, we obtain

$$\begin{aligned} &W(x) \sum_{j=k(x)}^{k(x)+1} |l_{jn}(x)| W^{-1}(x_{jn})(1+|x_{jn}|)^{-\alpha} \\ &\quad \sim (1+|x|)^{-\alpha} W(x) \sum_{j=k(x)}^{k(x)+1} |l_{jn}(x)| W^{-1}(x_{jn}) \\ &\quad \geqslant C_2(1+|x|)^{-\alpha}. \end{aligned}$$

In particular, if $\tilde{\Lambda}_n(x)$ contains the terms in the last sum, we obtain

$$\tilde{\Lambda}_n(x) \geqslant C_3(1+|x|)^{-\alpha}, \quad \text{for } x \in [x_{nn}, x_{1n}]. \tag{2.10}$$

This establishes the desired lower bound. Now (2.8) and (2.10) yield

$$\tilde{\Lambda}_n(x) \sim (1+|x|)^{-\alpha}, \quad \text{for } x \in [x_{nn}, x_{1n}]. \tag{2.11}$$

Part (a) is now a consequence of Lemma 2.2 and (2.11).

(b) Let $|x| \leqslant a_n(1+Ln^{-2/3})$, for $L > 0$ fixed. Then from (2.1) and the fact that for $x_{j+1,n} \geqslant 0$,

$$\psi_n(x_{jn}) \sim \psi_n(x_{j+1,n}), \quad \text{for } 1 \leqslant j \leqslant n-1,$$

(cf. (11.10) in [3]), we can choose C_1 as in Lemma 2.1(e) and set

$$\mathscr{S}_1 := \left\{ x_{jn} \colon |x - x_{jn}| \leqslant C_1 \frac{a_n}{n} \psi_n(x_{jn})^{-1/2} \right\}.$$

If $\mathscr{S}_1$ is not empty, then it is noteworthy that $\mathscr{S}_1$ contains some zeros of $p_n(\xi)$ closest to $x \in \mathbb{R}$. This is due to the inequality (2.4) of Lemma 2.1. Also from (2.1) and the symmetric property of the zeros of $p_n(W^2, x)$ about origin we infer that only for $x \in \mathbb{R}$ for which $|x| \leqslant a_n(1+Ln^{-2/3})$, $L > 0$ fixed, can we obtain zeros close to x. Now using Lemma 2.1(d) we obtain

$$\begin{aligned} \tilde{\Lambda}_n &\geqslant W(x) \sum_{x_{kn} \in \mathscr{S}_1} |l_{kn}(x)| W^{-1}(x_{kn})(1+|x_{kn}|)^{-\alpha} \\ &\sim \sum_{x_{kn} \in \mathscr{S}_1} a_n^{3/2}/n\psi_n(x_{jn})^{-1/4} \frac{|p_n(x)|}{|x - x_{jn}|} (1+|x_{kn}|)^{-\alpha} \\ &\sim \sum_{x_{kn} \in \mathscr{S}_1} (1+|x_{kn}|)^{-\alpha} \\ &\sim (1+|x|)^{-\alpha}, \end{aligned} \tag{2.12}$$

after using (2.6) and the arguments of part (a). Now (2.8) and (2.12) also yield

$$\tilde{\Lambda}_n(x) \sim (1+|x|)^{-\alpha}, \quad \text{for } |x| \leqslant a_n(1+Ln^{-2/3}), \tag{2.13}$$

and the proof of (b) follows from Lemma 2.2 and (2.12). This completes the proof of the Theorem. □

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